Finite Strip Method For Fixed Ended Plates

Dr. Mereen Hassan Fahmi

Technical College - Erbil

ABSTRACT

Finite strip method (FSM) is an effective method for analysis of slabs, slab bridges, slab girder bridges, box girder bridges and other different type of structures. In most previous studies the derivations of the stiffness matrix based on the assumption, that the boundary condition is hinged at both ends.

In this study the derivations are extended to determine the stiffness matrix and load vector for the fixed ended slab strip using the minimum total potential energy method. Two slab girder bridges are analyzed by finite strip method and the results are compared with the exact and finite element method solutions and showed good agreement.

KEYWORDS: Finite strip method, Fixed ended bridges, Harmonic function, Polynomial function and Slab girder bridges.

[illegible]

[illegible]

[illegible]

Received 4 Dec. 2005 **Accepted 15 June 2006**

NOTATIONS:

$[B_{bm}^{I}]$ = bending coefficient matrix.
$[D_{b}^{I}]$ = bending rigidity matrix.
E = modulus of elasticity of the material
$f_m(x)$ = polynomial function in (x) direction.
$\{k^{I}\}$ = curvature vector.
$[K_{bm}^{I}]$ = bending stiffness matrix.
$[K_{m}^{I}]$ = combine bending and in-plane stiffness matrix.
M_x, M_y = bending moments in x and y directions respectively.
M_{xy} = twisting moment in xy plane.
$\{P_{bm}^{I}\}$ = bending load vector.
q(x, y) = the loading function subjected on the plane of the plate.
t = thickness of the plate.
U = total strain energy.
V = total kinetic energy.
w(x, y) = displacement function in (z) direction.
w_{im} and w_{jm} = deflections at nodal line (i and j) for strip (m).
$\Phi_m(y)$ = Harmonic function in (y) direction.
υ = Poisson's ratio.
Π = total potential energy = U – V
θ_{im} and θ_{jm} = slopes at nodal line (i and j) for strip.

INTRODUCTION:

Many computer programs have been developed using the finite element method and used extensively in the analysis of box girder bridges and developed to include the elastic and inelastic of the material.

Finite strip method (FSM) is an effective method for analysis of different types of structures such as slabs, slab bridges, slab girder bridges, box girder bridges. The finite strip method first presented [1 and 2] to analyze the simply supported bridge deck structures. The solution based on an orthotropic plate theory and presented in the form of design curves in [3 and 4], using only the 1st term of the harmonic deflection function.

Finite strip method uses the advantages of both the orthotropic plate theory and finite element concept, and applied for both slabs and slab girder and box girder bridges. In this method, the displacement function assumed to be a combination of a one way polynomial function in transverse direction and harmonic function in longitudinal direction. The harmonic function satisfies the boundary conditions of both ends. In most of previous studies hinged boundary conditions are assumed at both ends to derive the stiffness matrix. This study presents the derivations of the stiffness matrix and load vector of the fixed ended boundary conditions. The harmonic function in longitudinal direction, which satisfies the boundary condition of the fixed ends has been used.

Petrolito and Golley [5, 6] considered the use of finite strip element method for the vibration analysis of thick plates, the shape function are obtained as the product of combined trigonometric and polynomial functions in the x-direction and polynomial function in y-direction. The simplest displacement approximation within the finite strip element is obtained by using a linear approximation in the y-direction. Higher order approximation in the y-direction can be achieved by using internal nodal lines. The global equations are derived in the usual manner of the finite element method and natural frequencies of vibration can be found by solution a linear Eigen values problem.

Azhari et al. [7], presented an analysis of the local buckling of composite laminated plates and folded plate assemblies subjected to arbitrary loading using the spline finite strip method. Because the spline finite strip method is fairly well-known in buckling analysis, its direct application to the local buckling of composite laminates has been more limited. The method is programmed to study the local buckling of laminated flat plated and L-sections.

Lirkov and morgenov [8] used the finite strip method to solve the fourth order boundary value problem (Bi-harmonic equation) for boundary conditions correspond to clamped, joint and free edges. The standard computational procedure reduced the problem to a set of linear systems of equations with seven diagonal matrices.

Lau et al. [9] presented a numerical analysis procedure for the 3D flutter analysis of bridges based on the spline finite strip method. The method has been extended to the area of bridge aerodynamic in wind engineering. The significant improvement of the presented formulation is that the effect of the spatial distribution of the aerodynamic forces on a bridge deck structure can be taken into account by distributing the aerodynamic forces over the cross section of the bridge deck.

DREIVATIONS:

Fig.(1) shows the deflection surface of fixed-fixed slab bridges under an arbitrary loading, the assumed displacement function must be general and flexible enough to represent the actual displacement filed of the slab bridge. The figure shows that the slab divided into rectangular strips spanning between the two fixed ends, the actual deformed surface can be represented by simple cosine displacement function in longitudinal direction and polynomial function in the perpendicular direction. The boundary conditions at the both ends must be satisfied by the assumed function, the displacement and the slope must be zero at the fixed ends. Fig.(2) show the displacement filed for typical fixed ended finite strip. The assumed function for a strip takes the form of a combination of cosine series in span wise direction and polynomial function of 3rd degree in the transverse direction.

$$w(x, y) = \Sigma_{m=1} f_m(x)\, \Phi_m(y) = \Sigma_{m=1} (A + Bx + Cx^2 + Dx^3 + \ldots)\, [1 - \cos(2\pi m y/a)] \text{ ----(1)}$$

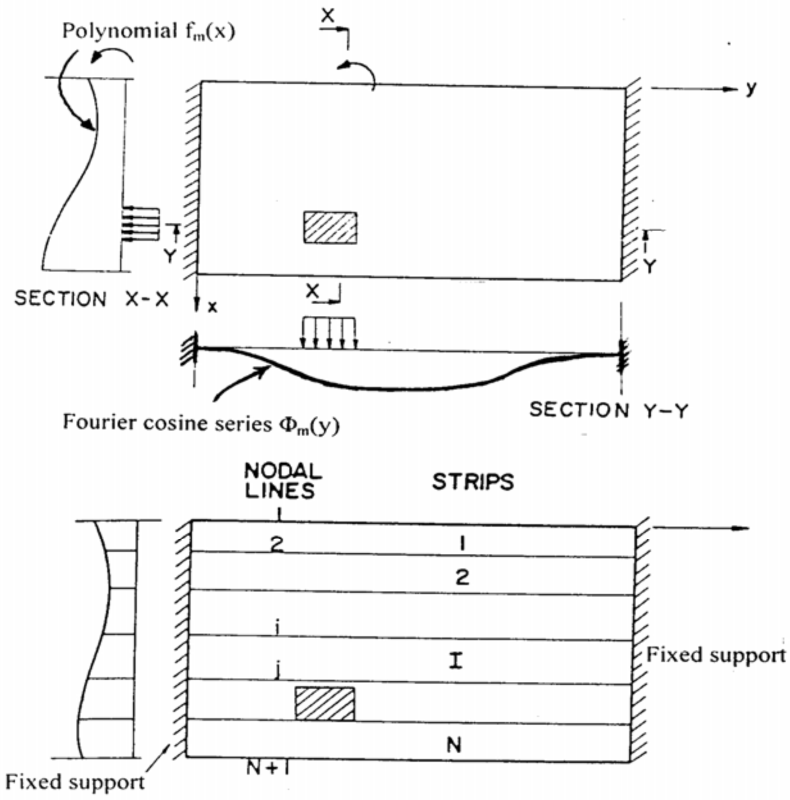

Fig.(1): Finite strip elements for fixed ended plates. [11]

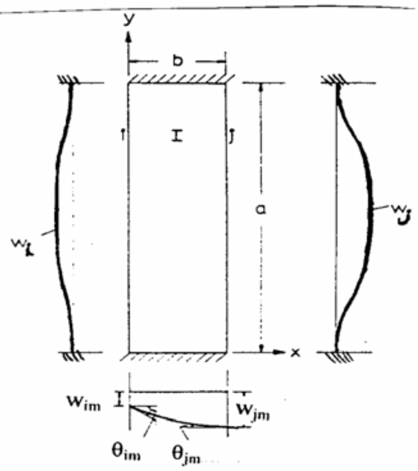

Fig.(2): Nodal displacements for fixed ended plate strip in bending.

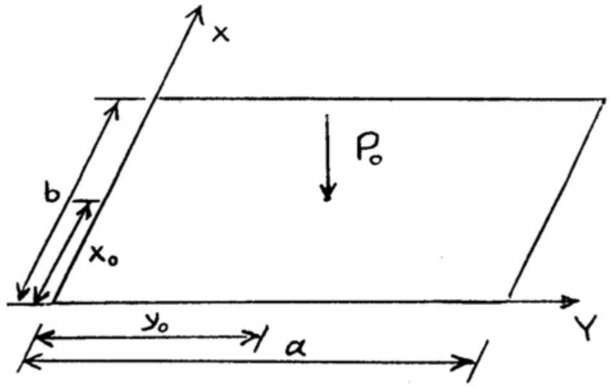

Fig.(3-a): Concentrated load.

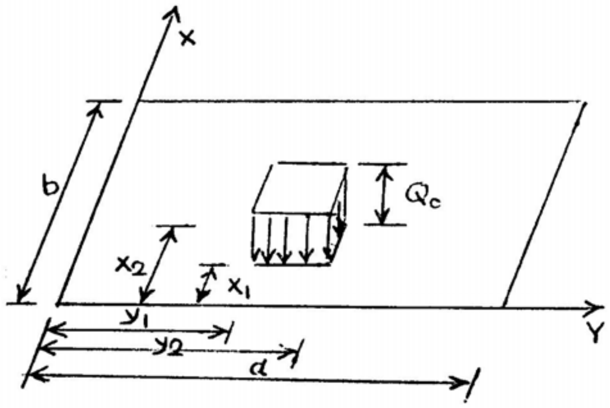

Fig.(3-b): Uniform patch load.

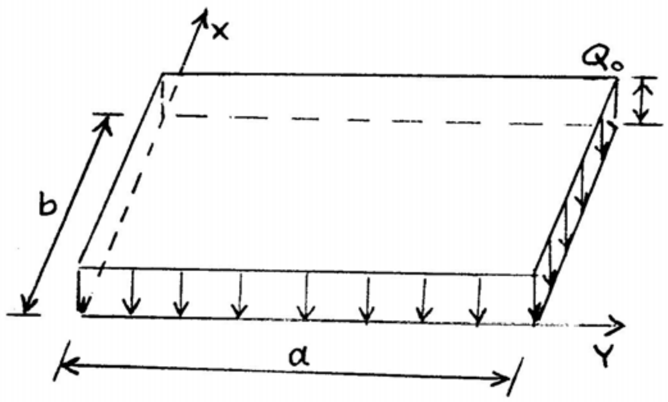

Fig.(3-c): Uniform distributed load.

Fig.(3): Transverse loading conditions.

where A, B, C, D, ……etc, are the polynomial coefficients, (m) represent the m^{th} harmonic term and (M) the number of harmonic terms to be used for the solution.
$\Phi_m(y) = 1 - \cos(2\pi my/a)$
Let $K_m = \pi m/a$, then $\Phi_m(y) = 1 - \cos(2K_m y)$
and $\Phi'_m(y) = 2\,K_m \sin(2\pi my/a)$
The above function satisfies the boundary function at fixed ended plate:
at y = 0 and a, $\Phi_m(y) = 0$ and $\Phi'_m(y) = 0$
The deflection amplitudes (w_{im} and w_{jm}) and slopes (θ_{im} and θ_{jm}) at the two nodal lines (i and j) for the fixed ended finite strip can be chosen the displacement and slope parameters at (x=0 and x=b), where (b) is the width of the strip element and (a) is the span of the strip element.

$$\theta_i = (\partial w/\partial x)_i = \Sigma_{m=1}\, \theta_{im}\, [1 - \cos(2K_m y)] \qquad \text{------(2)}$$
$$\theta_j = (\partial w/\partial x)_j = \Sigma_{m=1}\, \theta_{jm}\, [1 - \cos(2K_m y)] \qquad \text{------(3)}$$

It is necessary to use third order polynomial with four constants to incorporate the four unknowns (w_{im} , w_{jm} , θ_{im} and θ_{jm}).

$$f_m(x) = A + Bx + Cx^2 + Dx^3 \qquad \text{------(4)}$$
$$\partial f_m(x)/\partial x = B + 2Cx + 3Dx^2 \qquad \text{------(5)}$$

where A, B, C and D are arbitrary constants can be written in term of displacement unknown by applying the compatibility conditions:
at x = 0, $w_{im} = f_m(0)$ and $\theta_{im} = \partial f_m(0)/\partial x$
at x = b, $w_{jm} = f_m(b)$ and $\theta_{jm} = \partial f_m(b)/\partial x$
Substitute in above equations $f_m(x)$ and $f'_m(x)$ to obtain:
$A = w_{im}$
$B = \theta_{im}$

$$A + B\,b + C\,b^2 + D\,b^3 = w_{jm} \qquad \text{------(6)}$$
$$B + 2C\,b + 3D\,b^2 = \theta_{jm}$$

Rearrange the above equation in matrix form and solve for the constants (A, B, C and D):

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & b & b^2 & b^3 \\ 0 & 1 & 2b & 3b^2 \end{matrix} \quad \begin{matrix} A \\ B \\ C \\ D \end{matrix} \quad \begin{matrix} w_{im} \\ \theta_{im} \\ w_{jm} \\ \theta_{jm} \end{matrix} \qquad \text{------(7)}$$

or in compact form : $[\lambda]\ \{C_i\} = \{w_{bm}\}$; and $(C_i\} = [\lambda]^{-1}\ \{w_{bm}\}$
where $[\lambda]$ is the coefficient matrix, $\{C_i\}$ is the unknown vector and $\{w_{bm}\}$is the nodal displacement amplitudes vector in bending.
The resulting constants are:

$$C_{0i} = 1 - 3x^2/b^2 + 2x^3/b^3$$
$$C_{1i} = x - 2x^2/b + x^3/b^2$$
$$C_{2i} = 3x^2/b^2 - 2x^3/b^3 \qquad \text{------(8)}$$
$$C_{3i} = - x^2/b + x^3/b^2$$

The final displacement function w(x,y) can be written in matrix form as:

$$w(x,y) = \Sigma_{m=1}[C_{0i} \quad C_{1i} \quad C_{2i} \quad C_{3i}]\ [w_{im} \quad w_{jm} \quad \theta_{im} \quad \theta_{jm}]^T\ [1 - \cos(2K_m y)] \qquad \text{----(9)}$$

or simply:

$$w(x,y) = \Sigma_{m=1}[C_b^I]^T\ \{w_{bm}^I\}\ [1 - \cos(2K_m y)] \qquad \text{----------(10)}$$

where $[C_b^I]$ is the coefficient matrix.
The total potential energy of the strip in bending may be given as [10 and 11]:-

$$\Pi=\int_0\int_0\left[1/2[-M_x\partial^2 w/\partial x^2 - M_y\partial^2 w/\partial y^2 + 2M_{xy}\partial^2 w/\partial x\partial y] - q(x,y)w(x,y)\right]dx\ dy \qquad \text{----(11)}$$

In matrix form:

$$\Pi=\int_0\int_0\left[1/2[M_x\ M_y\ M_{xy}][-\partial^2 w/\partial x^2\ \ -\partial^2 w/\partial y^2\ \ 2\partial^2 w/\partial x\partial y]^T - q(x,y)w(x,y)\right]dx\ dy \qquad \text{--(12)}$$

or in compact form:

$$\Pi = \int_0\int_0 \left[1/2\ [M^I]^T\ \{k^I\} - q(x,y)\ w(x,y)\right]\ dx\ dy \qquad \text{----------(13)}$$

where M_x, M_y, M_{xy} are the bending moments in x and y directions and the twisting moment respectively.
$[M^I]$ is the bending moment vector, $\{k^I\}$ is the curvatures vector and q(x,y) is the loading function subjected on the plane of the plate. The 1st term represents the strain energy and the 2nd represents the kinetic energy of the strip element in bending
apply the 2nd derivatives of the displacement function w(x,y), the curvature vector can be written as the following:

$$\{k^I\}= [-\partial^2 w/\partial x^2 \quad -\partial^2 w/\partial y^2 \quad 2\partial^2 w/\partial x\partial y]^T = \Sigma_{m=1}\ [B_{bm}^I]\ \{w_{bm}^I\} \qquad \text{------------(14)}$$

where:

$$[B_{bm}^I] = \begin{matrix} -C''_{0i}\ \Phi_m(y) & -C''_{1i}\ \Phi_m(y) & -C''_{2i}\ \Phi_m(y) & -C''_{3i}\ \Phi_m(y) \\ -C_{0i}\ \Phi''_m(y) & -C_{1i}\ \Phi''_m(y) & -C_{2i}\ \Phi''_m(y) & -C_{3i}\ \Phi''_m(y) \\ 2C'_{0i}\ \Phi'_m(y) & 2C'_{1i}\ \Phi'_m(y) & 2C'_{2i}\ \Phi'_m(y) & 2C'_{3i}\ \Phi'_m(y) \end{matrix} \qquad \text{-----(15)}$$

The moment vector at any point in a strip can be also expressed in term of the curvatures as the following:

$$\{M^I\} = [M_x \quad M_y \quad M_{xy}]^T = [D_b^I]\ \{k^I\}= \Sigma_{m=1}\ [D_b^I]\ [B_{bm}^I]\ \{w_{bm}^I\} \qquad \text{--------(16)}$$

where $[D_b^I]$ is the rigidity matrix in bending =

$$\begin{matrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{matrix} \qquad \text{------(17)}$$

and D_x and D_y are the transverse and longitudinal flexural rigidity, D_{xy} is the torsional rigidity and D_1 is the coupling rigidity as defined in [12]
Substitute $\{M^I\}$ and $\{k^I\}$ in the equation of total potential energy to get:

$$\Pi=\Sigma_{m=1}\{w_{bm}^I)^T\int_0\int_0[1/2[B_{bm}^I]^T[D_b^I][B_{bm}^I]\{w_{bm}^I)-[C_b^I]^T q(x,y)[1-\cos(2K_m y)]]dxdy \quad (18)$$

The total potential energy is minimized with respect to the nodal displacements:

$$\partial\Pi/\partial\{w_{bm}^I\}=0 \quad \text{------}(19)$$

Applying the integration and then minimizing the total potential energy, the following equation is obtained:

$$\Sigma_{m=1}[K_{bm}^I]\ \{w_{bm}^I) = \{P_{bm}^I\} \quad \text{-------------}(20)$$

where: $[K_{bm}^I] = 1/2 \int_0\int_0\ [B_{bm}^I]^T[D_b^I]\ [B_{bm}^I]\ \ dx\ dy$ -------------(21)

and $\{P_{bm}^I) = \int_0\int_0\ [C_b^I]^T\ q(x,y)\ [1 - \cos(2K_m y)]\ \ dx\ dy$ -------------(22)

$[K_{bm}^I]$ is a (4 by 4) stiffness matrix for conventional strip in bending and $\{P_{bm}^I)$ is a (4 by 1) loading vector.
The resulting stiffness matrix will be:

$$[K_{bm}^I] = \begin{matrix} K1 & K2 & K3 & K4 \\ K2 & K5 & -K4 & K6 \\ K3 & -K4 & K1 & -K2 \\ K4 & K6 & -K2 & K5 \end{matrix} \quad \text{-------------}(23)$$

In compact form:

$$[K_{bm}^I] = \begin{matrix} [K_{ii}]_b & [K_{ij}]_b \\ [K_{ji}]_b & [K_{jj}]_b \end{matrix}$$

Sample integration of one element of the stiffness matrix (K_i)is:

$$K_1= \int_0\int_0\ [\ (b_{11}\ D_x +\ b_{21}\ D_1)\ b_{11} + (b_{11}\ D_1 + b_{21}\ D_y)\ b_{21} + b_{31}^2\ D_{xy}\ ]\ dx\ dy \quad \text{----------}(24)$$

Then applying the integration to get:

$K_1 = 18\ a/b^3\ D_x + 104/35\ ab\ K_m^4\ D_y + 48/5\ a/b\ K_m^2\ D_{xy} + 24/5\ a/b\ K_m^2\ D_1$ -------(25)

$K_2 = 9\ a/b^2\ D_x + 44/105\ ab^2\ K_m^4\ D_y + 4/5\ a\ K_m^2\ D_{xy} + 12/5\ a\ K_m^2\ D_1$ --------(26)

$K_3 = -\ 18\ a/b^3\ D_x + 36/35\ ab\ K_m^4\ D_y - 48/5\ a/b\ K_m^2\ D_{xy} - 24/5\ a/b\ K_m^2\ D_1$ --------(27)

$K_4 = 9\ a/b^2\ D_x - 26/105\ ab^2\ K_m^4\ D_y + 4/5\ a\ K_m^2\ D_{xy} + 2/5\ a\ K_m^2\ D_1$ --------(28)

$K_5 = 6\ a/b\ D_x + 8/105\ ab^3\ K_m^4\ D_y + 16/15\ ab\ K_m^2\ D_{xy} + 8/15\ ab\ K_m^2\ D_1$ --------(29)

$K_6 = 3\ a/b\ D_x - 2/35\ ab^3\ K_m^4\ D_y - 4/15\ ab\ K_m^2\ D_{xy} - 2/15\ ab\ K_m^2\ D_1$ -------(30)

In matrix form:

K_1	18	104/35	48/5	24/5	
K_2	9b	44/105b	4/5b	12/5b	$a/b^3\ D_x$

$$\begin{array}{lcccccl} K_3 = & -18 & 36/35 & -48/5 & -24/5 & ab\ K_m^4\ D_y & \\ K_4 & 9b & -26/105b & 4/5b & 2/5b & a/b\ K_m^2\ D_{xy} & \text{-----(31)} \\ K_5 & 6b^2 & 8/105b^2 & 16/15b^2 & 8/15b^2 & a/b\ K_m^2\ D_{xy} & \\ K_6 & 3b^2 & -2/35b^2 & -4/15b^2 & -2/15b^2 & & \end{array}$$

Load vectors for load types shown in Fig.(3):

1- Concentrated load (P_o) acts at coordinates (x_o and y_o):

$$\{P_{bm}^{\ I}\} = \begin{array}{l} Z_{im} \\ M_{im} \\ Z_{jm} \\ M_{jm} \end{array} = \begin{array}{l} 1 - 3x_o^2/b^2 + 2x_o^3/b^3 \\ x_o - 2x_o^2/b + x_o^3/b^2 \\ 3x_o^2/b^2 - 2x_o^3/b^3 \\ -x_o^2/b + x_o^3/b^2 \end{array} \quad P_o\,[1 - \cos(2K_m y)] \qquad \text{----(32)}$$

2- Patch load (Q_o) acts on the area bounded by two points (x1,y1) and (x2,y2):

$$\{P_{bm}^{\ I}\} = \begin{array}{l} X' - X'^3/b^2 + X'^4/2b^3 \\ X'^2/2 - 2\,X'^3/3b + X'^4/4b^2 \\ X'^3/b^2 - X'^4/2b^3 \\ -\ X'^3/3b + X'^4/4b^2 \\ - \end{array} \quad Q_o\,C_{my} \qquad \text{-------(33)}$$

where: $X' = x2 - x1$ and $C_{my} = (y2\text{-}y1) - 1/(2K_m)\ [\sin(2K_m y2) - \sin(2K_m y1)]$

3- Distributed load (Q_o) over the entire area of the strip:

$$\{P_{bm}^{\ I}\} = \begin{array}{c} b/2 \\ b^2/12 \\ b/2 \\ -b^2/12 \end{array} \quad Q_o\,[\,a - 1/(2K_m)\,\sin(2K_m\,a)] \qquad \text{-------(34)}$$

The total equation of the load – displacement relationship becomes:

$$\Sigma_{m=1}\,[K_m^{\ I}]\ \{w_m^{\ I}\} = \{P_m^{\ I}\} \qquad \text{---------(35)}$$

where: $[K_m^{\ I}]$ is a (4 by 4) total stiffness matrix for the finite strip in bending analysis. Two slab girder bridges shown in Fig.(4) are solved by finite strip method and compared with the finite element method (FEM) as shown below:

Application1: The bridge shown in Fig.(4) is divided into three elements and four nodes, span=10m, υ=0.25 and P=44.5 kN. The nodal displacements determined by (FSM) are given in Table (1). The central displacement and central bending moment are compared with finite element method and given in Table(2).

Table (1): The nodal displacement determined by (FSM).

Nodal line	Central displacement α Pa3/EI
1	0.00558
2	0.0056
3	0.0056
4	0.00558

Table(2): Comparison of (FSM) and (FEM) results.

Method of analysis	Average central deflection α Pa3/EI	Central bending moment β P a
Finite strip method (FSM)	0.0056	0.1254
Exact method (Analytical)	0.0053	0.125

Application2: The bridge shown in Fig. (4) divided into seven elements and eight nodes, span=10m, υ=0.25 and P=445 kN. The nodal displacements determined by (FSM) are given in Table (3). The central displacement and central bending moment are compared with finite element method and given in Table (4).

Table (3): The nodal displacement determined by (FSM).

Nodal line	Central displacement α Pa3/EI (FSM)	Central displacement α Pa3/EI (FEM)
1	0.00808	0.00861
2	0.00813	0.00866
3	0.00837	0.00895
4	0.00846	0.00906

Table (4): Comparison of (FSM) and (FEM) results.

Method of analysis	Central bending moment β P a
Finite strip method (FSM)	0.12
Finite element method (FEM)	0.125

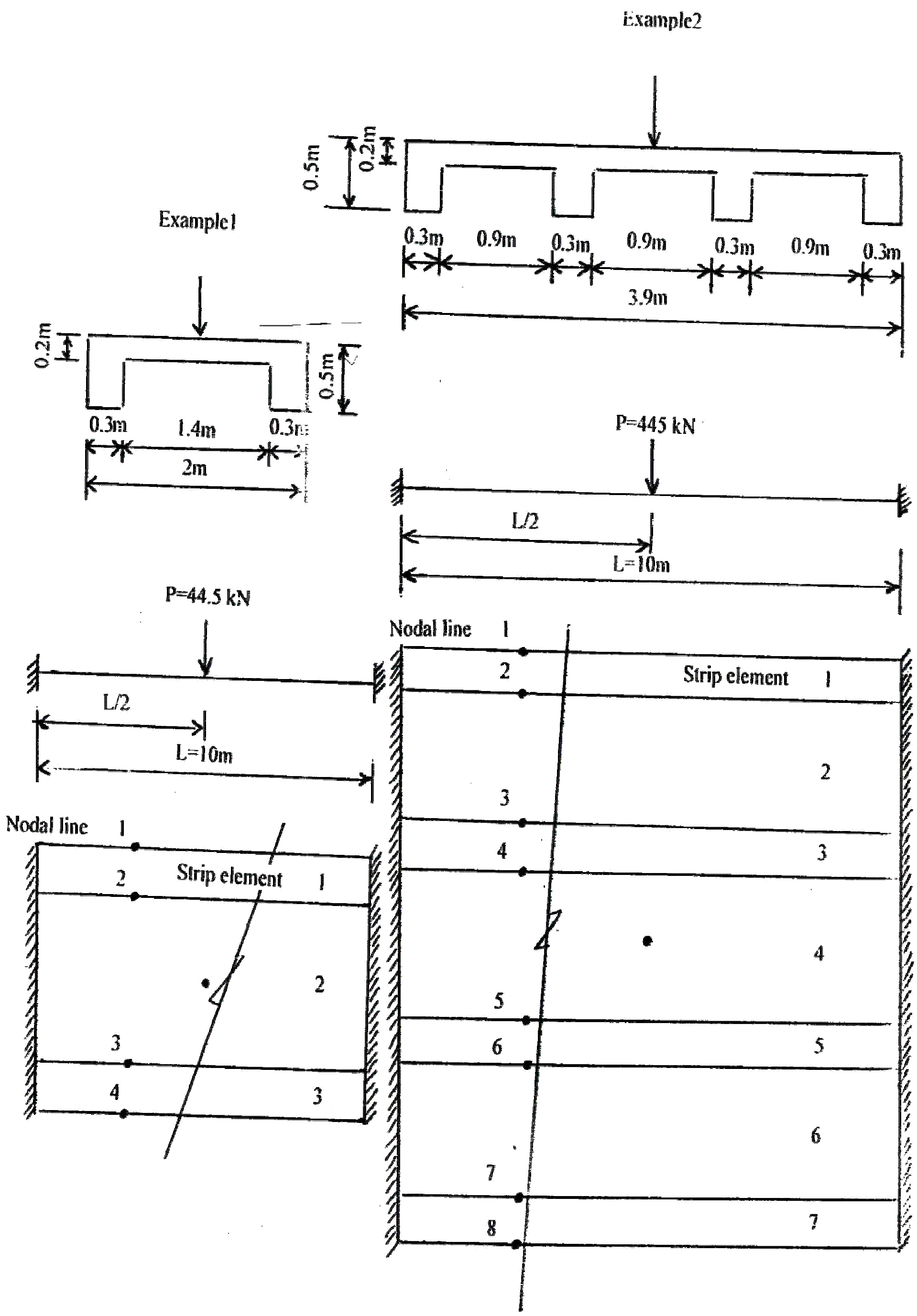

Fig.(4): Slab girder bridges analyzed by FSM.

CONCLUSIONS:

1-Finite strip method is an effective method for analysis of different types of structures such as slabs, slab bridges, slab girder bridges, box girder bridges and its uses the advantages of both the orthotropic plate theory and finite element concept

2-The derivation of the equations and programming in the finite strip method is simpler than finite element method. Finite strip method needs smaller number of strips and smaller size of total stiffness matrix, thus shorter computing time in comparison with finite element method.

3-Results obtained by finite strip method showed good agreement with that obtained by finite element method.

REFERENCES:

1-Cheung, Y.K., "Finite strip method analysis of elastic slabs", Proc. ASCE, Vol.94, No.EM6, December, PP. 1365-1378, 1968.

2-Cheung, Y.K., "Finite strip method analysis in the analysis of elastic plates with two opposite simply supported ends", Proc. Inst. Civil Engineers, Vol.40, December, PP. 1-7, 1968.

3-Morice, P.B., Little, G. and Rowe, R.E., "Design curves for the effect of concentrated loads on concrete bridge decks", Publication of Cement and Concrete Association, 1956.

4-Cusens, A.R. and Pama, R.P., "Bridge deck analysis", John Wiley and Sons, London, 1975.

5-Petrolito, J. and Golley, B.W., "Vibration of Thick Plates Using Finite Strip Elements", Computational Techniques and Applications Conference and Workshops – CTAC99, Secpember 20 – 24, The Australian National University, Canberra, ACT, Australia, 1999.

6-Petrolito, J. and Golley, B.W., "Vibration of Thick Plates Using Finite Strip Elements", ANZIAM Journal, Vol. 42, PP.C1137 – C1155, 25 December, 2000.

7-Azhari, M., Abdollahian, M. and Bradford, M.A., "Local Buckling of Composite Laminated Plate Assemblies Using the Spline Finite Strip Method", Advances in Structural Engineering, Vol.3, No. 2, PP. 173 – 178,1 April, 2000.

8-Lirkov, I. and Morgenov, S., "Finite Strip Method for Bi-Harmonic Equation", PDE Methods in Applied Mathematics and Image Processing, Sponsored by the Alexander Von Humboldt Foundation within the Stability Pact for Southeastern Europe, Sunny Beach, Bulgaria, September 6 – 10, 2004.

9-Lau, D.T., Cheung, M.S. and Cheng, S.H., "3D Flutter Analysis of Bridges by Spline Finite Strip Method", Journal of Structural Engineering, Vol.126, No.10, PP. 1246 – 1254, October, 2000.

10-Timoshinko, S. and Woinowsky-Krieger, S., "Theory of Plates and Shells", 2nd edition, Mc-Graw Hill book Company, 1984.

11-Loo, Y.C. and Cusens, A.R., "Finite Strip Method in Bridge Engineering", A Viewpoint Publication, 1978.

12-Timoshinko, S. and Goodier J.N.,"Theory of Elasticity", 3rd edition, Mc-Graw Hill Kogakusha Ltd., 1970.